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GKS and other inequalities for Ising model with random interactions

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Abstract. We give a generalisation of the well known GKS inequalities and other correlation inequalities (derived by Thompson and by Krinsky and Emery) to the case of quenched correlation functions of the general Ising model with random ferromagnetic interactions.

1. Introduction

In the last few years there has been an increasing interest, both experimental and theoretical, in random magnetic systems. This paper contains an extension of GKS inequalities (Griffiths 1971) to a class of such systems, and an improved form of some inequalities derived by Thompson (1971) and by Krinsky and Emery (1974), which appeared useful for Ising ferromagnets. Namely, we consider the general Ising ferromagnet given by the following Hamiltonian:

$$H=-\sum_{P\subset\Lambda} J_P\sigma_P, \qquad \sigma_P=\prod_{i\in P}\sigma_i, \qquad \sigma_i=\pm 1, \tag{1a}$$

where Λ is a finite set of points, and the interaction constants J_P are assumed to be statistically independent, non-negative random variables, i.e.

$$J_P\geqslant 0, \qquad \rho=\prod_{P\subset\Lambda}\rho(J_P), \tag{1b}$$

where ρ is a joint probability density and $\rho(J_P)$ is a probability density of a random variable J_P.

For the cases in which J_P are fixed (not random) many correlation inequalities have been proved. We are interested here in GKS inequalities (Ginibre 1970):

$$\langle\sigma_R\rangle\geqslant 0, \qquad R\subset\Lambda, \tag{2a}$$

$$\langle\sigma_R\sigma_S\rangle-\langle\sigma_R\rangle\langle\sigma_S\rangle\geqslant 0, \tag{2b}$$

$$k\in R, \qquad \langle\sigma_R\rangle\leqslant\sum_{\substack{S\ni k\\ S\subset\Lambda}}\tau(S)\langle\sigma_R\sigma_S\rangle, \tag{2c}$$

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and in a Thompson inequality (see Thompson 1971, where it is derived for the nearest neighbour Ising ferromagnet, and Jędrzejewski 1978, where a general proof is given):

$$k \in R, \qquad \langle \sigma_R \rangle \leqslant \tau\Big(\sum_{\substack{S \ni k \\ S \subset \Lambda}} J_S \langle \sigma_R \sigma_S \rangle \Big), \tag{3}$$

and also in the Krinsky and Emery inequality (Krinsky and Emery 1974, Jędrzejewski 1978):

$$k \in R, \qquad \langle \sigma_R \rangle \leqslant \tau\Big(\sum_{\substack{S \ni k \\ S \subset \Lambda}} \tau^{-1}(\langle \sigma_{RS} \rangle \tau(S)) \Big) \tag{4}$$

where $\tau(S) \equiv \tanh(\beta J_S)$, $\beta = 1/k_B T$ and $\langle \ldots \rangle$ denotes the canonical average (in the original proof of Krinsky and Emery the summation is over all S such that $S \cap R \neq \varnothing$). In the following §§ 2 and 3 we prove analogous inequalities for quenched correlation functions, i.e. for quantities $\overline{\langle \sigma_R \rangle}$ where the bar denotes the average with respect to a distribution of all interaction constants J_P. The main tools used in our proofs are the following lemmas.

Lemma 1.1. (Falk 1977) Let Γ be a finite set of points, $\{f_i(\,.\,), i \in \Gamma\}$ a set of functions of a random variable x, $f_A(\,.\,) = \Pi_{i \in A} f_i(\,.\,)$ and $E[f_A]$ denote an expectation value. If, for all $A \subset \Gamma$, $f_A(\,.\,)$ are non-decreasing functions of x with non-negative expectation values, then

$$E[f_A] \geqslant \prod_{i \in A} E[f_i].$$

Lemma 1.2. (Jędrzejewski 1977) Consider the model with the Hamiltonian equations (1*a*) and (1*b*) and a correlation function $\langle \sigma_R \rangle$ for some $R \subset \Lambda$. If there exist a set Q such that $\varnothing \neq Q \subset R$ and $J_P = 0$ if $P \cap Q \neq \varnothing$, then

$$\langle \sigma_R \rangle = \tau(R).$$

In §4 we make comments on the ordering of quenched magnetisation bounds which follow from the considered inequalities and on the inequalities derived by Falk and Gehring (1975).

2. GKS inequalities for quenched correlation functions

In this section we show that GKS inequalities established for correlation functions of the general Ising ferromagnet (Ginibre 1970) are valid also for quenched correlation functions.

Lemma 2.1. The following inequalities:

$$\overline{\langle \sigma_R \rangle} \geqslant 0, \qquad R \subset \Lambda, \tag{5a}$$

$$\overline{\langle \sigma_{RS} \rangle} - \overline{\langle \sigma_R \rangle \langle \sigma_S \rangle} \geqslant 0, \tag{5b}$$

$$k \in R, \qquad \overline{\langle \sigma_R \rangle} \leqslant \sum_{S \ni k} \overline{\tau(S) \langle \sigma_{RS} \rangle}, \tag{5c}$$

hold for the model defined by equations (1*a*) and (1*b*).

Proof. (5*a*) is obvious and (5*b*) is an immediate consequence of lemma 1.1 in § 1, so only (5*c*) needs a proof. We start from the equality (Thompson 1971):

$$\langle\sigma_R\rangle(1+\tau(S)\langle\sigma_S\rangle_S) = \langle\sigma_R\rangle_S + \tau(S)\langle\sigma_{RS}\rangle_S, \tag{6}$$

where the subscript S denotes that $J_S = 0$.

Let $E_S[\,.\,]$ and $E_{\Lambda\setminus S}[\,.\,]$ denote the averages with respect to J_S and with respect to all the other interaction constants respectively. The average value of the right-hand side of (6) is equal to

$$\overline{\langle\sigma_R\rangle_S} + \overline{\tau(S)\langle\sigma_{RS}\rangle_S} = \overline{\langle\sigma_R\rangle_S} + E_{\Lambda\setminus S}[E_S[\tau(S)]E_S[\langle\sigma_{RS}\rangle_S]] = \overline{\langle\sigma_R\rangle_S} + \overline{\tau(S)}\,\overline{\langle\sigma_{RS}\rangle_S} \tag{7a}$$

(because $E_S[\langle\sigma_{RS}\rangle_S] = \langle\sigma_{RS}\rangle_S$) while the average value of the left-hand side of (6) is bounded below by

$$\overline{\langle\sigma_R\rangle} + \overline{\langle\sigma_R\rangle\tau(S)\langle\sigma_S\rangle_S} \geq \overline{\langle\sigma_R\rangle} + \overline{\langle\sigma_R\rangle}\,\overline{\tau(S)}\,\overline{\langle\sigma_S\rangle_S}, \tag{7b}$$

where we used lemma 1.1 of §1. (7*a*) and (7*b*) give us the following inequality (Falk 1977):

$$\overline{\langle\sigma_R\rangle}(1+\overline{\tau(S)}\,\overline{\langle\sigma_{RS}\rangle}) \leq \overline{\langle\sigma_R\rangle_S} + \overline{\tau(S)}\,\overline{\langle\sigma_{RS}\rangle_S}; \tag{8a}$$

hence

$$\overline{\langle\sigma_R\rangle} \leq \overline{\langle\sigma_R\rangle_S} + \overline{\tau(S)}\,\overline{\langle\sigma_{RS}\rangle}. \tag{8b}$$

Now, we iterate (8*b*) over all sets S containing some point, say k, of a set R:

$$S_i \ni k, \qquad \overline{\langle\sigma_R\rangle} \leq \overline{\langle\sigma_R\rangle_{S_1\ldots S_n}} + \overline{\tau(S_1)}\,\overline{\langle\sigma_{RS_1}\rangle} + \sum_{i=2}^{n} \overline{\tau(S_i)}\,\overline{\langle\sigma_{RS_i}\rangle_{S_1\ldots S_i}} \tag{9}$$

Because of lemma 1.2 of §1, $\overline{\langle\sigma_R\rangle_{S_1\ldots S_n}} = \overline{\tau(R)}$ and because of the second GKS inequality (2*b*) $\overline{\langle\sigma_{RS_i}\rangle_{S_1\ldots S_i}} \leq \overline{\langle\sigma_{RS_i}\rangle}$, so from (9) we obtain (5*c*).

3. Thompson inequality and Krinsky and Emery inequality for quenched correlation functions

Thompson (1971) derived, for the magnetisation of the translationally invariant Ising ferromagnet with nearest neighbour interactions, an inequality which gives a bound for the magnetisation equal to the solution of the mean field approximation (MFA) equation. This inequality was subsequently strengthened and extended to the case of general Ising ferromagnet by Krinsky and Emery (1974). We present here both inequalities for quenched correlation functions.

Lemma 3.1. The following inequalities are valid for quenched correlation functions of the general Ising ferromagnet (equations (1*a*) and (1*b*)) for the Thompson inequality:

$$\overline{\langle\sigma_R\rangle} \leq \tau\Big(\sum_{S\ni k} \overline{J_S}\,\overline{\langle\sigma_{RS}\rangle}\Big), \qquad k\in R \tag{10}$$

and for the Krinsky and Emery inequality:

$$\overline{\langle\sigma_R\rangle} \leq \tau\Big(\sum_{S\ni k} \tau^{-1}(\overline{\langle\sigma_{RS}\rangle}\,\overline{\tau(S)})\Big). \tag{11}$$

Proof. Proofs of both inequalities (10) and (11) generally go along the same lines as the proof of (5*c*). First let us prove the inequality (10). We apply the second GKS inequality (2*b*) in the following form:

$$\langle\sigma_S\rangle_S \geqslant \langle\sigma_{RS}\rangle_S\langle\sigma_R\rangle_S \tag{12a}$$

and the inequality

$$\langle\sigma_{RS}\rangle_S\tau(S) \leqslant \tau(J_S\langle\sigma_{RS}\rangle_S) \tag{12b}$$

to equation (6), which results in

$$\langle\sigma_R\rangle + \langle\sigma_R\rangle\langle\sigma_R\rangle_S\tau(J_S\langle\sigma_{RS}\rangle_S) \leqslant \langle\sigma_R\rangle_S + \tau(J_S\langle\sigma_{RS}\rangle_S). \tag{13}$$

Averaging (13) over interaction constants J_P and taking into account lemma 1.1 of § 1, we obtain

$$\overline{\langle\sigma_R\rangle} + \overline{\langle\sigma_R\rangle}\,\overline{\langle\sigma_R\rangle_S\tau(J_S\langle\sigma_{RS}\rangle_S)} \leqslant \overline{\langle\sigma_R\rangle_S} + \overline{\tau(J_S\langle\sigma_{RS}\rangle_S)}. \tag{14}$$

Because of the concavity of the hyperbolic tangent for positive argument, we have Hölder's inequality (Hölder 1889) $\overline{\tau(x)} \leqslant \tau(\bar{x})$. Therefore

$$\overline{\langle\sigma_R\rangle}[1 + \langle\sigma_R\rangle_S\tau(\bar{J}_S\overline{\langle\sigma_{RS}\rangle_S})] \leqslant \overline{\langle\sigma_R\rangle_S} + \tau(\bar{J}_S\overline{\langle\sigma_{RS}\rangle_S})$$

or equivalently

$$\overline{\langle\sigma_R\rangle} \leqslant \tau[\tau^{-1}(\overline{\langle\sigma_R\rangle_S}) + \bar{J}_S\overline{\langle\sigma_{RS}\rangle_S}]. \tag{15}$$

Now let $k \in R$. After the iteration of (15) over all sets S such that $S \ni k$ and making use of lemma 1.2 of § 1 and of (2*b*) we obtain (10).

In proving (11) one only omits (12*b*), hence instead of (14) one gets

$$\overline{\langle\sigma_R\rangle}(1 + \overline{\langle\sigma_R\rangle_S}\,\overline{\langle\sigma_{RS}\rangle_S}\,\overline{\tau(S)}) \leqslant \overline{\langle\sigma_R\rangle_S} + \overline{\langle\sigma_{RS}\rangle_S\tau(S)}$$

or equivalently

$$\overline{\langle\sigma_R\rangle} \leqslant \tau[\tau^{-1}(\overline{\langle\sigma_R\rangle_S}) + \tau^{-1}(\overline{\langle\sigma_{RS}\rangle_S}\,\overline{\tau(S)})]. \tag{16}$$

For $k \in R$ the iteration over all $S \ni k$, lemma 1.2 of § 1 and (2*b*) give us (11).

4. Remarks

In the case of a translationally invariant Ising model with z nearest-neighbour, random interactions J and a positive magnetic field h, the inequality (10) reduces to the following one:

$$\overline{\langle\sigma\rangle} \leqslant \tau(z\beta\bar{J}\overline{\langle\sigma\rangle} + \beta h), \tag{17a}$$

where $\overline{\langle\sigma\rangle}$ is the one-site, quenched correlation functon, i.e. the quenched magnetisation. Thus we have found that the exact quenched magnetisation $\overline{\langle\sigma\rangle}$ is bounded by the positive solution of the equation

$$m = \tau(z\beta\bar{J}m + \beta h). \tag{17b}$$

This solution which we denote by $\overline{m}_{\mathrm{MFA}}$ may be regarded as a quenched magnetisation in the mean field approximation. In the same model the inequality (11) which is stronger than (10) becomes

$$\overline{\langle\sigma\rangle} \leq \tau[\beta h + z\tau^{-1}(\overline{\langle\sigma\rangle}\,\overline{\tau(\beta J)})]. \tag{18a}$$

Therefore $\overline{\langle\sigma\rangle}$ is bounded by the positive solution, denoted by $\overline{m}_{\mathrm{KE}}$, of the equation

$$m = \tau[\beta h + z\tau^{-1}(m\overline{\tau(\beta J)})], \tag{18b}$$

which is less than $\overline{m}_{\mathrm{MFA}}$, so we have that

$$\overline{\langle\sigma\rangle} \leq \overline{m}_{\mathrm{KE}} \leq \overline{m}_{\mathrm{MFA}}. \tag{19}$$

Falk (1977) proved that the quenched magnetisation in the Bethe approximation, which is given by

$$\overline{m}_{\mathrm{Be}} = \tau[\beta h + z\tau^{-1}(m^*\overline{\tau(\beta J)})] \tag{20a}$$

where m^* is a positive solution of the equation

$$m = \tau[\beta h + (z-1)\tau^{-1}(m\overline{\tau(\beta J)})] \tag{20b}$$

is the upper bound for the exact, quenched magnetisation. By comparison of (18*b*) and (20*b*) we see that $m^* < \overline{m}_{\mathrm{KE}}$, hence we have got the following ordering of the quenched magnetisation bounds:

$$\overline{\langle\sigma\rangle} \leq \overline{m}_{\mathrm{Be}} \leq \overline{m}_{\mathrm{KE}} \leq \overline{m}_{\mathrm{MFA}}. \tag{21}$$

Finally, let us mention that a result of Falk and Gehring (1975), that a quenched correlation function of two spins does not exceed a corresponding correlation function, considered as a function of averaged interaction constants, has its counterpart in the model given by equations (1*a*) and (1*b*). Namely, because of the inequality $\partial^2\langle\sigma_R\rangle/\partial^2 J_P \leq 0$ (Ginibre 1970) we have that

$$\overline{\langle\sigma_R\rangle} \leq \langle\sigma_R\rangle_{\bar{J}} \tag{22}$$

(Jensen's inequality), where $\langle\ldots\rangle_{\bar{J}}$ denotes the canonical average with the Hamiltonian equations (1*a*) and (1*b*) in which J_P are replaced by their mean values $\bar{J}_P$.

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